Gauging of Nonlinearly Realized Symmetries

I. N. McArthur

Department of Physics, The University of Western Australia
Nedlands, Australia. 6907

ABSTRACT

A representation of a subgroup H of a finite-dimensional group G can be used to induce a nonlinear realization of G. If the nonlinearly realized symmetry is gauged, then the BRST charge can be related by a similarity transformation to the BRST charge for the gauged linear realization of H (plus a cohomologically trivial piece). It is shown that the relation between the two BRST charges is a reflection of the fact that they can be interpreted geometrically as expressions for the exterior derivative on G relative to two different bases, and an explicit expression for the generator of the similarity transformation is obtained. This result is applied in an infinite-dimensional setting, where it yields the similarity transformation used by Ishikawa and Kato to prove the equivalence of the Berkovits-Vafa superstring with the underlying bosonic string theory.

1 Introduction

Recently, nonlinear realizations of superconformal algebras have appeared in string theory in two separate guises. Gato-Rivera and Semikhatov [1] and Bershadsky et al [2] have shown that the field content of noncritical string theories is generally adequate to admit a nonlinearly realized twisted N=2 superconformal symmetry. The BRST current (or an improved version of it) is one of the odd generators of the superconformal algebra, so that the nonlinearly realized symmetry encodes the BRST structure of the theory in some way. The other guise is in the Berkovits-Vafa construction [3] in which N=0 bosonic string theories can be considered as N=1 superstring theories with a special choice of background. Central to this construction is an embedding of the Virasoro algebra into a nonlinearly realized N=1 superconformal algebra. Both of these manifestations of nonlinearly realized superconformal symmetries can be found in higher N superstrings and in W-strings [2, 6, 7, 8].

Amplitudes computed for the Berkovits-Vafa superstring are equivalent to those for the underlying bosonic string [3], which is a reflection of the fact that the BRST cohomologies of the two theories are identical [4]. A heuristic explanation for this has been advanced by Polchinski [5]. The basic tenet is that one can promote a linear realization of a symmetry based on a group H to a nonlinearly realized symmetry based on a larger group G by the introduction of extra fields ("Goldstone bosons" parameterizing G/H). Gauging this nonlinearly realized symmetry (without kinetic terms for the gauge fields) gives a theory equivalent to the original one with the linearly realized symmetry gauged, in that the gauge degrees of freedom associated with G/H precisely cancel the extra degrees of freedom introduced in extending the symmetry. In the case of the Berkovits-Vafa embeddings, the fermionic primary fields (b_1, c_1) of conformal weight $(\frac{3}{2}, -\frac{1}{2})$ which are introduced to extend a linearly realized critical Virasoro symmetry to a nonlinearly realized critical N=1 superconformal algebra are effectively cancelled by the bosonic ghosts (β, γ) of the superstring, leaving a critical bosonic string theory.

For a related problem (the embedding of a Kac-Moody algebra into an N=1 super Kac-Moody algebra), Figueroa-O'Farrill has carried out an analysis of the above mechanism at a cohomological level [11]. A simpler approach is adopted by Ishikawa and Kato [9] who demonstrate by explicit construction that there is a similarity transformation which maps the BRST charge $Q_{N=1}$ of the Berkovits-Vafa superstring to that of the bosonic string $Q_{N=0}$ (plus a cohomologically trivial topological charge Q_{TOP})[1],

$$e^R\, Q_{N=1}\, e^{-R} = Q_{N=0} + Q_{TOP}. \tag{1}$$

[1] This construction has been extended to the case of the embedding of N=1 superstrings as special backgrounds of N=2 superstrings in [10], and to hierarchies of w strings and super w strings in [8].

The existence of this transformation guarantees the equivalence of the BRST cohomologies of the two theories [11]. Although Ishikawa and Kato state the form of the charge R in (1), they provide no general means for determining it. The homological methods of [11] are claimed to be generally applicable, but given the simplicity of the approach adopted by Ishikawa and Kato, it is important to obtain a better understanding of the nature of the charge R in cases more general than that treated in [9].

The aim of this paper is to set in place a formalism which allows the calculation of the charge R in similarity transformations of the type (1) relating the BRST charge of a gauged nonlinear extension of a symmetry to that for the underlying linearly realized symmetry. The problem is tackled only at the level of nonlinear realizations of finite dimensional symmetry algebras (as opposed to infinite dimensional symmetry algebras such as Kac-Moody algebras and superconformal algebras). However, it is shown that at least in the case of the Berkovits-Vafa construction, the results derived in the finite dimensional setting can be applied in the infinite dimensional setting without "quantum correction." It would be interesting to know whether this is a general phenomenon.

This absence of quantum corrections to R is to be contrasted with the situation for the nonlinearly realized superconformal algebra itself, where only part of the structure of this infinite dimensional algebra can be deduced by applying the standard formalism for nonlinear realizations of finite dimensional algebras [12, 13]. The quantum corrections have to be determined "by hand" using the requirement that the nonlinearly realized generators close under operator product expansion. In fact, (1) provides a systematic method for deriving the quantum corrections to the nonlinear realization – knowledge of R and the form of $Q_{N=0}$ and Q_{TOP} allows $Q_{N=1}$ to be determined, from which the nonlinearly realized generators can be read. These generators must include the quantum corrections as the construction (1) ensures that $Q^2_{N=1} = 0$. Again, if R can be determined fully in other situations by application of the finite dimensional results derived in this paper, then quantum corrections to nonlinearly realized symmetries could be obtained by similar means. An example is the twisted N=2 superconformal algebra underlying noncritical string theories. The "classical" form of this algebra can be found by Hamiltonian reduction [2], but it suffers quantum corrections. These can be determined by applying quantum Hamiltonian reduction [14], which is a fairly complicated procedure (although recent advances [15] allow a systematic approach). If the results of this paper can be applied, considerable simplification may follow.

The content of the paper is as follows. In §2, formalism relating to induced representations of finite-dimensional groups is introduced, and expressions for the nonlinearly realized generators of a group G induced from a linear realization of a subgroup H are obtained. The BRST charge Q_G for the gauged nonlinearly realized symmetry is constructed in §3, and it is shown that there exists a charge R bilinear

in the ghost fields such that

$$e^R Q_G e^{-R} = Q_H + Q_0, \tag{2}$$

where Q_H is the BRST charge associated with the gauged linearly realized symmetries and Q_0 is a cohomologically trivial charge. *In particular, it will be seen that the nilpotent charges Q_G and $Q_H + Q_0$ can be interpreted geometrically as representations of the exterior derivative on G relative to different bases.* The expression obtained for R in the finite-dimensional case is applied in §4 to the infinite-dimensional case of the Berkovits-Vafa embedding, where it is shown to yield the charge R in (1) found by Ishikawa and Kato [9].

It should be stressed that the work in this paper simply aims to obtain an algorithm which allows the charge R in (2) to be computed, and the mathematical formalism adopted is tailored to this requirement. It is quite probable that these results can be formulated more elegantly in more sophisticated mathematical language.

2 Induced Representations

A representation (or linear realization) of a group H can be used to induce a nonlinear realization of a group G containing H as a subgroup. At a more mathematical level, the induced representation is on sections of a vector bundle over G/H associated to G, which can be considered as a principal H-bundle over G/H (see, for example, [16]). Induced representations occur widely in physics. An example is spontaneous symmetry breaking in quantum field theory, where the unbroken generators of the symmetry are realized nonlinearly and the broken generators are realized linearly on the Goldstone modes associated with the spontaneous symmetry breaking [17, 18].

Initially we will restrict attention to finite-dimensional groups. Let T_a ($a = 1, 2, \cdots, \dim \mathrm{G}$) denote the generators of G, chosen so that T_i ($i = 1, 2, \cdots, \dim \mathrm{H}$) are the generators of the subgroup H, with the remaining generators X_α ($\alpha = 1, 2, \cdots, \dim \mathrm{G} - \dim \mathrm{H}$) spanning the complement of **h** in **g** (where **h** and **g** are the Lie algebras of H and G respectively). Assuming that the coset G/H is symmetric, then the structure constants ${f_{ab}}^c$ of g are determined by

$$\begin{aligned}
[T_i, T_j] &= {f_{ij}}^k T_k \\
[T_i, T_\alpha] &= {f_{i\alpha}}^\beta T_\beta \\
[T_\alpha, T_\beta] &= {f_{\alpha\beta}}^k T_k
\end{aligned} \tag{3}$$

(summation over repeated indices). Elements of G can be parameterized (at least

in a neighbourhood of the identity) in the form[2]

$$g(y,\xi) = e^{y^i T_i}\, e^{\xi^\alpha X_\alpha}. \tag{4}$$

The coordinates ξ^α parameterize a slice through the identity locally isomorphic to H/G, and the y^i parameterize the H-orbits through points on this slice.

The Lie algebra valued one-form $g^{-1}dg$ on G can be decomposed as

$$g^{-1}dg = \omega^a T_a, \tag{5}$$

where the ω^a form a basis of left-invariant one-forms on G. The Maurer-Cartan equation is

$$d\omega^a = -\frac{1}{2} f_{bc}{}^a\, \omega^b \wedge \omega^c. \tag{6}$$

Introducing one-forms ϕ^i on the H-orbits by

$$e^{-y^i T_i} d\, e^{y^i T_i} = \phi^i(y) T_i,$$

then (4) yields

$$\begin{aligned} g^{-1}dg &= e^{-\xi^\alpha X_\alpha}\, \phi^i T_i\, e^{\xi^\alpha X_\alpha} + e^{-\xi^\alpha X_\alpha} d\; e^{\xi^\alpha X_\alpha} \\ &= \phi^i (T_i - A_i{}^\alpha X_\alpha + \frac{1}{2!} A_i{}^\alpha A_\alpha{}^j T_j - \cdots) \\ &\quad + d\xi^\alpha (X_\alpha - \frac{1}{2!} A_\alpha{}^i T_i + \frac{1}{3!} A_\alpha{}^i A_i{}^\beta X_\beta - \cdots) \end{aligned}$$

where

$$A_\alpha{}^i = \xi^\beta f_{\beta\alpha}{}^i, \qquad A_i{}^\alpha = \xi^\beta f_{\beta i}{}^\alpha. \tag{7}$$

Thus it follows from (5) that

$$\omega^i = \phi^j\, (1 + \frac{1}{2!} A^2 + \frac{1}{4!} A^4 + \cdots)_j{}^i - d\xi^\beta\, (\frac{1}{2!} A + \frac{1}{4!} A^3 + \cdots)_\beta{}^i \tag{8}$$

and

$$\omega^\alpha = -\phi^j\, (A + \frac{1}{3!} A^3 + \cdots)_j{}^\alpha + d\xi^\beta\, (1 + \frac{1}{3!} A^2 + \frac{1}{5!} A^4 + \cdots)_\beta{}^\alpha. \tag{9}$$

Let Y_a denote a basis of left-invariant vector fields on G dual to the basis ω^a of left-invariant one-forms, $\omega^a(Y_b) = \delta^a_b$, and let η_i be vector fields on the H-orbits dual to the one-forms ϕ^i, $\phi^i(\eta_j) = \delta^i{}_j$. Since

$$d\phi^i = -\frac{1}{2} f_{jk}{}^i \phi^j \wedge \phi^k, \tag{10}$$

the vector fields η_i provide a realization of the subalgebra H of G,

$$[\eta_i, \eta_j] = f_{ij}{}^k \eta_k. \tag{11}$$

[2] We adopt the (unconventional) right coset parameterization to obtain a nonlinear realization which acts by right derivatives of ξ^α.

Similarly the Maurer-Cartan equation (6) implies

$$[Y_a, Y_b] = f_{ab}{}^c Y_c. \tag{12}$$

The vector fields Y_a can be decomposed relative to the basis $(\eta_i, \frac{\partial}{\partial \xi^\alpha})$ of vector fields,

$$Y_a = Y_a{}^i \eta_i + Y_a{}^\alpha \frac{\partial}{\partial \xi^\alpha}.$$

It follows from (8), (9) and $\omega^a(Y_b) = \delta^a_b$ that[3]

$$\begin{aligned}
Y_i{}^j &= \delta_i^j \\
Y_i{}^\beta &= A_i^\beta \\
Y_\alpha{}^\beta &= \Big[(1 + \frac{1}{3!}A^2 + \frac{1}{5!}A^4 + \cdots)^{-1} (1 + \frac{1}{2!}A^2 + \frac{1}{4!}A^4 + \cdots)\Big]_\alpha{}^\beta \\
Y_\alpha{}^j &= \Big[(1 + \frac{1}{3!}A^2 + \frac{1}{5!}A^4 + \cdots)^{-1} (\frac{1}{2!}A + \frac{1}{4!}A^3 + \cdots)\Big]_\alpha{}^j .
\end{aligned} \tag{13}$$

Using these results,

$$\begin{aligned}
Y_i &= \eta_i + A_i{}^\alpha \frac{\partial}{\partial \xi^\alpha} \\
Y_\alpha &= \frac{\partial}{\partial \xi^\alpha} + \frac{1}{2} A_\alpha{}^i \eta_i + \frac{1}{3} A_\alpha{}^i A_i^\beta \frac{\partial}{\partial \xi^\beta} - \frac{1}{24} A_\alpha{}^i A_i^\beta A_\beta{}^j \eta_j + O(A^4),
\end{aligned} \tag{14}$$

where higher order terms can be computed using (13). Since the left-invariant vector fields Y_a satisfy the commutation relations (12), it then follows that (14) is a nonlinear realization of the Lie algeba of G. If the η_i, which satisfy (11), are replaced by a representation of the generators of H on some vector space with coordinates v, then (14) provides a nonlinear realization of G on the space with coordinates (v, ξ). As expected for a realization induced from a representation of H, the subgroup H is realized linearly on this space (the A are linear in ξ).

3 Gauging the Induced Representation

Gauging the induced representation of G leads to the introduction of a pair of fermionic ghosts (b_a, c^a) for each generator of G. These satisfy the anticommutation relation $\{b_a, c^b\} = \delta_a{}^b$. The corresponding BRST charge is

$$Q_G = c^a Y_a + \frac{1}{2} c^a T_a^{(gh)}, \tag{15}$$

where the ghost charges

$$T_a^{(gh)} = -f_{ab}{}^c c^b b_c \tag{16}$$

[3] A more compact notation could be introduced, for example $Y_\alpha{}^\beta = \big[A(\sinh A)^{-1} \cosh A\big]_\alpha{}^\beta$, $Y_\alpha{}^j = \big[(\sinh A)^{-1}(\cosh A - 1)\big]_\alpha{}^j$.

satisfy $[T_a^{(gh)}, T_b^{(gh)}] = f_{ab}{}^c T_c^{(gh)}$. By construction the BRST charge Q_G is nilpotent, $\{Q_G, Q_G\} = 0$.

As is well known, the BRST charge Q_G can be interpreted geometrically as an exterior derivative on the manifold G. Relative to the basis ω^a if left-invariant one-forms on G, the exterior derivative on G is

$$d = \omega^a Y_a. \tag{17}$$

In Q_G, the role of the one-forms ω^a is played by the ghost fields c^a, with $c^a Y_a$ reproducing the exterior derivative of functions on G. The term $\frac{1}{2} c^a T_a^{(gh)}$ in Q_G reproduces the action of the exterior derivative on forms, as determined by (6).

Relative to the basis $(\phi^i, d\xi^\alpha)$ for one-forms on G, the exterior derivative takes the form

$$d = \phi^i \eta_i + d\xi^\alpha \frac{\partial}{\partial \xi^\alpha}. \tag{18}$$

A corresponding nilpotent charge $\tilde{Q}_G$ can be realized in terms of fermionic ghosts $(\tilde{b}_a, \tilde{c}^a)$ (with $\{\tilde{b}_a, \tilde{c}^b\} = \delta_a{}^b$) by

$$\tilde{Q}_G = \tilde{c}^i \eta_i - \frac{1}{2} \tilde{c}^i (f_{ij}{}^k \tilde{c}^j \tilde{b}_k) + \tilde{c}^\alpha \frac{\partial}{\partial \xi^\alpha}. \tag{19}$$

With the identification $(\tilde{c}^i, \tilde{c}^\alpha) \to (\phi^i, d\xi^\alpha)$, the terms linear in $\tilde{c}^a$ reproduce the action of the exterior derivative on functions on G. The term trilinear in the ghosts reproduces the action of the exterior derivative on ϕ^i, as given by (10). The absence of any trilinear terms involving the ghost fields $(\tilde{b}_\alpha, \tilde{c}^\alpha)$ is a reflection of the fact that $d(d\xi^\alpha) = 0$.

The two expressions (17) and (18) for the exterior derivative on G are simply expressions relative to different bases for vector fields on G (and the corresponding dual bases of one-forms), namely

$$\begin{aligned} d &= \omega^i Y_i + \omega^\alpha Y_\alpha \\ &= \omega^a Y_a{}^i \eta_i + \omega^a Y_a{}^\alpha \frac{\partial}{\partial \xi^\alpha}. \end{aligned}$$

On the other hand, the ghost fields playing the role of one-forms in the the nilpotent charges (15) and (19) corresponding to (17) and (18) respectively are "passive" in nature, in that the action of the exterior derivative on forms has to be reproduced by terms trilinear in the ghosts. Consequently, the change of basis which relates Q_G and $\tilde{Q}_G$ must be implemented "actively" by a transformation of the ghost fields. If we identify c^a and $\tilde{c}^a$, then the terms in Q_G and $\tilde{Q}_G$ not involving ghost generators are $c^a Y_a = c^a Y_a{}^i \eta_i + c^a Y_a{}^\alpha \frac{\partial}{\partial \xi^\alpha}$ and $c^i \eta_i + c^\alpha \frac{\partial}{\partial \xi^\alpha}$ respectively. The "change of basis" involved in going from the first expression to the second is the transformation $c^a \to c^b (Y^{-1})_b{}^a$. Since this is a linear transformation on the vector space of ghosts c^a, it

can be achieved by a similarity transformation generated by a bilinear R in the b_a and c^a,

$$e^R c^a e^{-R} = c^b (Y^{-1})_b{}^a. \tag{20}$$

As will be demonstrated later, the similarity transformation also achieves the desired change of basis for the terms trilinear in ghost fields, so that

$$e^R Q_G e^{-R} = \tilde{Q}_G.$$

This is precisely the relation (2) described in the introduction, with $Q_H = c^i \eta_i - \frac{1}{2} c^i (f_{ij}{}^k c^j b_k)$ and $Q_0 = c^\alpha \frac{\partial}{\partial \xi^\alpha}$.

To establish the above results, we first determine the form of the generator R of the transformation. Expressing R in the form

$$R = c^a K_a{}^b b_b, \tag{21}$$

then the anticommutation relation $\{b_a, c^b\} = \delta_a{}^b$ yields $[R, c^a] = c^b K_b{}^a$, so that

$$e^R c^a e^{-R} = c^b (e^K)_b{}^a.$$

Thus we require $(e^K)_b{}^a = (Y^{-1})_b{}^a$. Letting

$$Y^{-1} = 1 + Z,$$

then the matrix K is expressed in terms of the matrix Z by

$$K = \ln(1+Z) = Z - \frac{1}{2} Z^2 + \frac{1}{3} Z^3 - \frac{1}{4} Z^4 + \cdots . \tag{22}$$

The matrices $Y^{-1} = 1 + Z$ can be read from the expressions (8) and (9) using $\omega^a = \phi^i (Y^{-1})_i{}^a + d\xi^\alpha (Y^{-1})_\alpha{}^a$. The results are

$$\begin{aligned}
Z_i{}^j &= (\frac{1}{2!} A^2 + \frac{1}{4!} A^4 + \cdots)_i{}^j = [\cosh A - 1]_i{}^j \\
Z_\alpha{}^j &= -(\frac{1}{2!} A + \frac{1}{4!} A^3 + \cdots)_\alpha{}^j = [A^{-1}(1 - \cosh A)]_\alpha{}^j \\
Z_i{}^\beta &= -(A + \frac{1}{3!} A^3 + \cdots)_i{}^\beta = -[\sinh A]_i{}^\beta \\
Z_\alpha{}^\beta &= (\frac{1}{3!} A^2 + \frac{1}{5!} A^4 + \cdots)_\alpha{}^\beta = [A^{-1} \sinh A - 1]_\alpha{}^\beta.
\end{aligned} \tag{23}$$

These expressions allow K and hence R to be determined to any desired order in ξ^α using (22) and (21). To $O(\xi^3)$,

$$\begin{aligned}
R &= -c^i A_i{}^\alpha b_\alpha - \frac{1}{2} c^\alpha A_\alpha{}^i b_i + \frac{1}{4} c^i A_i{}^\alpha A_\alpha{}^j b_j \\
&\quad - \frac{1}{2} c^\alpha A_\alpha{}^i A_i{}^\beta b_\beta + \frac{1}{24} c^\alpha A_\alpha{}^i A_i{}^\beta A_\beta{}^j b_j + O(\xi^4),
\end{aligned} \tag{24}$$

where ${A_i}^\alpha$ and ${A_\alpha}^i$ are given by (7).

So far, the charge R has been constructed to ensure

$$(e^R c^a e^{-R}) Y_a = c^i \eta_i + c^\alpha \frac{\partial}{\partial \xi^\alpha}.$$

To prove that $e^R Q_G e^{-R} = \tilde{Q}_G$, it is necessary to show that

$$(e^R c^a e^{-R})(e^R Y_a e^{-R} - Y_a) + \frac{1}{2} e^R (c^a T_a^{(gh)}) e^{-R} = -\frac{1}{2} c^i {f_{ij}}^k c^j b_k.$$

Note that $e^R Y_a e^{-R}$ is nonzero as R is ξ dependent and the nonlinearly realized generators (14) contain ξ derivatives.

Using $e^R c^a e^{-R} = c^b {(Y^{-1})_b}^a$ and $e^R b_a e^{-R} = {Y_a}^b b_b$ and the definition (16) of $T_a^{(gh)}$, one obtains

$$\frac{1}{2} e^R (c^a T_a^{(gh)}) e^{-R} = -\frac{1}{2} c^d c^e b_f {(Y^{-1})_d}^a {f_{ab}}^c {(Y^{-1})_e}^b {Y_c}^f.$$

If we denote the basis $(\eta_i, \frac{\partial}{\partial \xi^\alpha})$ of vector fields by V_a, then the only nonvanishing commutator is $[V_i, V_j] = {f_{ij}}^k V_k$. Using $Y_a = {Y_a}^b V_b$ and (12),

$$\begin{aligned} -\frac{1}{2} c^d c^e \quad b_f \quad {(Y^{-1})_d}^a {f_{ab}}^c {(Y^{-1})_e}^b {Y_c}^f = & -\frac{1}{2} c_i {f_{ij}}^k c^j b_k \\ & + \frac{1}{2} c^d c^e b_f \Big({(Y^{-1})_d}^c (V_e . {Y_c}^f) - {(Y^{-1})_e}^c (V_d . {Y_c}^f) \Big), \end{aligned}$$

where $V_e . {Y_c}^f$ denotes the Lie derivative of the function ${Y_c}^f$ with respect to the vector field V_e (the "directional" derivative). Thus it remains to show that

$$c^d c^e b_f {(Y^{-1})_d}^c (V_e . {Y_c}^f) = -(e^R c^a e^{-R})(e^R Y_a e^{-R} - Y_a). \tag{25}$$

Using $e^R c^a e^{-R} = c^b {(Y^{-1})_b}^a$ and $Y_a = {Y_a}^b V_b$, the right hand side of this expression is $-c^a (e^R V_a e^{-R} - V_a)$. Now, since by (21), $V_a . R = c^b (V_a . {K_b}^c) b_c$, and $[c^a {B_a}^b b_b, R] = c^a {[B, K]_a}^b b_b$ for any matrix B, it follows that

$$\begin{aligned} e^R V_a e^{-R} - V_a &= -V_a . R + \frac{1}{2!} [V_a . R, R] - \frac{1}{3!} [[V_a . R, R], R] + \cdots \\ &= c^b \left(-V_a . K + \frac{1}{2!} [V_a . K, K] - \frac{1}{3!} [[V_a . K, K], K] + \cdots \right)_b^{\ c} b_c \\ &= c^b (e^K V_a e^{-K} - V_a)_b^{\ c} b_c \\ &= c^b (Y^{-1} V_a Y - V_a)_b^{\ c} b_c. \end{aligned}$$

This yields

$$-(e^R c^a e^{-R})(e^R Y_a e^{-R} - Y_a) = c^b c^a (Y^{-1} V_a Y - V_a)_b^{\ c} b_c$$

which is precisely the required result (25). So we have indeed established that

$$e^R Q_G e^{-R} = \tilde{Q}_G.$$

This has been explicitly checked by hand to $O(\xi^3)$ using expression (24) for R.

4 An Infinite Dimensional Application

Although the results of the previous section were derived in the context of finite-dimensional groups, it will be shown here that the result (1) of Ishikawa and Kato showing the triviality of the Berkovits-Vafa embedding is an example of this construction in an infinite-dimensional context.

As mentioned in the introduction, the essence of the Berkovits-Vafa construction is that by starting with a c=26 Virasoro algebra (determining a critical string theory), it is possible to extend this to a nonlinearly realized critical (c=15) super-Virasoro algebra by the inclusion of fermionic primary fields[4] (b_1, c_1) of conformal weight $(\frac{3}{2}, -\frac{1}{2})$. Specifically, if T denotes the generator of the c=26 Virasoro algebra, then the nonlinearly realized c=15 super-Virasoro algebra has generators $\tilde{T}(z)$ and $\tilde{G}(z)$ given by [3]

$$\begin{aligned}\tilde{T}(z) &= T(z) - \frac{3}{2}{:}b_1\partial c_1{:}(z) - \frac{1}{2}{:}\partial b_1 c_1{:}(z) + \frac{1}{2}\partial^2(c_1\partial c_1)(z)\\ \tilde{G}(z) &= b_1(z) + c_1 T(z) + {:}b_1 c_1 \partial c_1{:}(z) + \frac{5}{2}\partial^2 c_1(z).\end{aligned} \tag{26}$$

Gauging the nonlinearly realized super-Virasoro algebra (26) results in the introduction of fermionic ghosts (b, c) of conformal weight $(2, -1)$ and bosonic ghosts (β, γ) of conformal weight $(\frac{3}{2}, -\frac{1}{2})$. In terms of the Berkovits-Vafa construction in which the critical bosonic string is regarded as a critical superstring in a special background, these are the standard superconformal ghosts. The corresponding BRST charge is

$$\begin{aligned}Q_{N=1} &= \oint dz (c\tilde{T} - \frac{1}{2}\gamma\tilde{G} - {:}c{:}b\partial c{::} - \frac{1}{4}b\gamma^2\\ &\quad + \frac{1}{2}\partial c{:}\beta\gamma{:} - c{:}\beta\partial\gamma{:})(z),\end{aligned} \tag{27}$$

and is nilpotent due to the fact that the central charge c=15 of the super-Virasoro algebra is the critical value.

As shown by explicit construction in [9], there exists a charge R bilinear in ghost fields such that

$$e^R Q_{N=1} e^{-R} = Q_{N=0} + Q_{TOP},$$

where $Q_{N=0} = \oint dz\, (cT - {:}c{:}b\partial c{::})(z)$ is the BRST charge of the critical Virasoro algebra generated by T and $Q_{TOP} = -\frac{1}{2}\oint dz\, b_1\gamma(z)$ is a BRST charge associated with a cohomologically trivial topological sector composed of the fields (b_1, c_1) and (β, γ). The charge R is given by Ishikawa and Kato as

$$R = \oint dz c_1(\frac{1}{2}\gamma\beta - 3\partial c\beta - 2c\partial\beta - \frac{1}{2}\partial c_1{:}cb{:} + \frac{1}{4}\partial c_1{:}\gamma\beta{:})(z). \tag{28}$$

[4]These fields were denoted (b,c) in [13] where no confusion with the standard ghosts of string theory could arise. We adopt the more standard [3] notation (b_1, c_1) in this paper.

We will show that this result can be obtained by a naive application of the results of the previous section.

To extend the results of §3 to this case, two generalizations must be made. The first is that we are dealing with a superalgebra, and the analogues of the generators X_α associated with the coset G/H are odd generators (namely, the modes of the supercurrent). Correspondingly the analogues of the parameters ξ^α are Grassmann-odd and the analogues of the ghosts b_α and c^α are bosonic. Careful analysis of the steps in the the previous section leading to the expression for R shows that it is unchanged in the case of a superalgebra *provided* that the bosonic ghosts b_α and c^α satisfy the commutation relation $[b_\alpha, c^\beta] = \delta_\alpha{}^\beta$ (the fermionic ghosts b_i and c^i still satisfy $\{b_i, c^j\} = \delta_i{}^j$). This differs in sign from the usual convention for bosonic ghosts, and must be borne in mind when comparing with the results of Ishakawa and Kato.

The second extension is from a finite-dimensional superalgebra **g** and subalgebra **h** to an infinite dimensional one, namely the super-Virasoro algebra with a critical (c=26) Virasoro subalgebra. Thus the even generators T_i of **h** are are replaced by the modes L_n of the Virasoro algebra (along with a central term which will be denoted $\mathbf{1}$), the role of the odd generators X_α of **g** is played by the modes G_m of the supercurrent[5], and the commutation relations (3) are replaced by

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{13}{6}(m^3 - m)\,\delta_{m+n,0}\,\mathbf{1} \\
[L_m, G_n] &= (\frac{m}{2} - n)G_{m+n} \\
\{G_m, G_n\} &= 2L_{m+n} + \frac{26}{3}(m^2 - \frac{1}{4})\,\delta_{m+n,0}\,\mathbf{1}.
\end{aligned} \tag{29}$$

The analogues of the nonlinearly realized generators Y_i and Y_α of G in (14) are the modes $\tilde{L}_m$ and $\tilde{G}_m$ of the generators $\tilde{T}(z)$ and $\tilde{G}(z)$ of the nonlinearly realized super-Virasoro algebra (26). As detailed in [13], the role of the Grassmann-odd parameter ξ^α and its derivative $\frac{\partial}{\partial\xi^\alpha}$ are played respectively by the modes $(c_1)_{-m}$ and $(b_1)_m$ of the fermionic fields $c_1(z)$ and $b_1(z)$, the anticommutation relation $\{\frac{\partial}{\partial\xi^\alpha}, \xi^\beta\} = \delta_\alpha{}^\beta$ being replaced by $\{(b_1)_m, (c_1)_n\} = \delta_{m+n,0}$. The expressions (26) for the nonlinearly realized generators of the super-Virasoro algebra *cannot* be obtained by a simple translation of the finite-dimensional results (14); there are nontrivial quantum corrections which must be added to the generators to obtain a closed super-Virasoro algebra (with c=15) [13].

To form the BRST charge associated with the nonlinearly realized algebra (26), the fermionic ghosts b_i and c^i in (15) must be replaced by the modes b_m and c_{-m} of the superstring ghosts $b(z)$ and $c(z)$ respectively (with $\{b_m, c_n\} = \delta_{m+n,0}$). Special care is required in the identification of the bosonic ghosts b_α and c^α with the modes of their infinite-dimensional analogues β and γ. The required identification is of

[5] We work in the Ramond sector.

(b_α, c^α) with $(2\beta_m, -\frac{1}{2}\gamma_{-m})$. The sign is necessary to ensure that the commutation relation $[b_\alpha, c^\beta] = \delta_\alpha^\beta$ maps to the standard bosonic ghost commutation relation $[\beta_m, \gamma_n] = -\delta_{m+n,0}$. The factors of 2 are associated with the normalization of the terms in the BRST current: it contains $c^\alpha Y_\alpha$ in the finite-dimensional case (15), but $-\frac{1}{2}\gamma_{-m}\tilde{G}_m$ in the infinite-dimensional case (27).

The final ingredients required are the analogues of the matrices $A_\alpha{}^i = \xi^\beta f_{\beta\alpha}{}^i$ and $A_i{}^\alpha = \xi^\beta f_{\beta i}{}^\alpha$. To avoid confusion of indices in the infinite-dimensional case (where $i \to m$ and $\alpha \to m$), it is convenient to denote $A_i{}^\alpha = \xi^\beta f_{\beta i}{}^\alpha$ by $B_i{}^\alpha$. Then it follows by comparison of (3) and (29) that

$$A_m{}^p = 2(c_1)_{m-p}, \qquad A_m{}^{\mathbf{1}} = \frac{26}{3}(m^2 - \frac{1}{4})\,(c_1)_m$$

(where the index 1 is associated with the central term $\mathbf{1}$ in the super-Virasoro algebra) and

$$B_m{}^p = (p - \frac{3}{2}m)\,(c_1)_{m-p}, \qquad B_{\mathbf{1}}{}^p = 0.$$

Putting these results together, the direct translation of the expression (24) to the infinite-dimensional case is

$$\begin{aligned} R &= \frac{1}{2}\,(c_1)_{-m-p}\gamma_m b_p + 2(p + \frac{3}{2})\,(c_1)_{-m-p}c_m\beta_p \\ &\quad + \frac{1}{2}(-p + \frac{m}{2})\,(c_1)_p(c_1)_{-m-n-p}c_m b_n + \frac{1}{4}(p + \frac{n}{3})\,(c_1)_p(c_1)_{-m-n-p}\gamma_m\beta_n \\ &\quad + \frac{1}{8}(p - \frac{2n}{3})\,(c_1)_{-m-p}(c_1)_{-n+p}(c_1)_{n-q}\gamma_m b_q + O(c_1^4). \end{aligned} \tag{30}$$

In fact, the term cubic in c_1 and the $O(c_1^4)$ terms vanish. This is because they contain $(ABA)_m{}^p$, which can be seen to vanish by cycling the mode sums involving the $(c_1)_n$. Using the standard mode decompositions

$$\begin{aligned} b_1(z) &= \sum_n (b_1)_n z^{-n-\frac{3}{2}},\ c_1(z) = \sum_n (c_1)_n z^{-n+\frac{1}{2}},\ b(z) = \sum_n b_n z^{-n-2}, \\ c(z) &= \sum_n c_n z^{-n+1},\ \beta(z) = \sum_n \beta_n z^{-n-\frac{3}{2}},\ \gamma(z) = \sum_n \gamma_n z^{-n+\frac{1}{2}}, \end{aligned}$$

the result (30) can be put in the form

$$R = \oint dz\, c_1(\frac{1}{2}\gamma b - 3\partial c\beta - 2c\partial\beta - \frac{1}{2}\partial c_1 cb + \frac{1}{4}\partial c_1\gamma\beta)(z). \tag{31}$$

The expression (28) can be obtained from the above result simply by normal ordering the cb and $\gamma\beta$ terms. In particular, *no quantum corrections are necessary to obtain the correct expression (28) for R.*

5 Conclusion

In this paper, a general formula for the charge R in (2) has been obtained in a finite-dimensional setting. It demonstrates that if extra degrees of freedom are introduced to induce a nonlinear realization of G from a representation of a subgroup H, then gauging the nonlinearly realized symmetry "undoes" the construction [5]. A naive extension of the results to the case of the nonlinear realization of the N=1 superconformal algebra induced from a representation of the critical Virasoro subalgebra has been shown to reproduce the result (1) of Ishikawa and Kato [9].

As has already been noted, the structure of the nonlinearly realized super-Virasoro algebra (26) can in part be derived using the standard theory of nonlinear realizations [12, 13] (in fact, the results in [13] can be reproduced by applying the identifications in §4 to the formulas (14)). However, there are quantum corrections which must be included to make the algebra close. A knowledge of R allows these to be computed, as (1) can be inverted to yield $Q_{N=1}$:

$$Q_{N=1} = e^{-R}(Q_{N=0} + Q_{TOP})e^{R}$$

where, from (19), $Q_{N=0} = \oint dz(cT - :c{:}b\partial c{::})(z)$ and $Q_{TOP} = -\frac{1}{2}\oint dz\, b_1\gamma(z)$. The nonlinearly realized generators $\tilde{T}(z)$ and $\tilde{G}(z)$ can then be read from $Q_{N=1}$ as the coefficients of the terms linear in $c(z)$ and $-\frac{1}{2}\gamma(z)$. They are guaranteed to include the quantum corrections, as only the generators with these present will give a nilpotent BRST charge $Q_{N=1}$: the nilpotency of $Q_{N=1}$ follows by (1) from that of $Q_{N=0} + Q_{TOP}$, which in turn follows because $T(z)$ is the generator of a Virasoro algebra with c=26.

It would be interesting to know whether the finite-dimensional results of §3 for R can be applied in general infinite-dimensional settings without the need for quantum correction. If so, the procedure outlined in the above paragraph would allow a systematic derivation of "quantum nonlinear realizations" for superconformal algebras with N>1. An example is the nonlinearly realized twisted N=2 superconformal algebra underlying noncritical string theories [2], which at present must be obtained by quantum Hamiltonian reduction [2, 14].

Acknowledgement

I wish to thank Dr J. McCarthy for discussions related to this work.

References

[1] B. Gato-Rivera and A.M. Semikhatov, Phys. Lett. **B293** (1992) 72.

[2] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Nucl. Phys. **B401** (1993) 304.

[3] N. Berkovits and C. Vafa, Mod. Phys. Lett. **A9** (1994) 653.

[4] J.M. Figueroa-O'Farrill, Phys. Lett. **B321** (1994) 344.

[5] J. Polchinski, *What is String Theory?*, ITP Santa Barbara preprint NSF-ITP-94-97, hep-th/9411028.

[6] F. Bastianelli, N. Ohta and J.L. Petersen, Phys. Rev. Lett. **73** (1994) 1199.

[7] N. Berkovits, M. Freeman and P. West, Phys. Lett. **325** (1994) 63.

[8] H. Kunitomo, M. Sakaguchi and A. Tokura, Int. J. Mod. Phys. **A10** (1995) 827; Prog. Theor. Phys. **92** (1994) 1019.

[9] H. Ishikawa and M. Kato, Mod. Phys. Lett. **A9** (1994) 725.

[10] N. Ohta and J.L. Petersen, Phys. Lett. **B325** (1994) 67.

[11] J.M. Figueroa-O'Farrill, Nucl. Phys. **B432** (1994) 404.

[12] H. Kunitomo, Phys. Lett. **B343** (1995) 144.

[13] I.N. McArthur, Phys. Lett. **B342** (1995) 94.

[14] K. Ito and H. Kanno, Mod. Phys. Lett. **A9** (1994) 1377; 3063.

[15] J. de Boer and T. Tjin, Commun. Math. Phys. **158** (1993) 485; **160** (1994) 317.

[16] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I,* (Springer-Verlag, Berlin, 1972), Chapter 5.

[17] S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2239.

[18] C.G. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2247.